

# Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

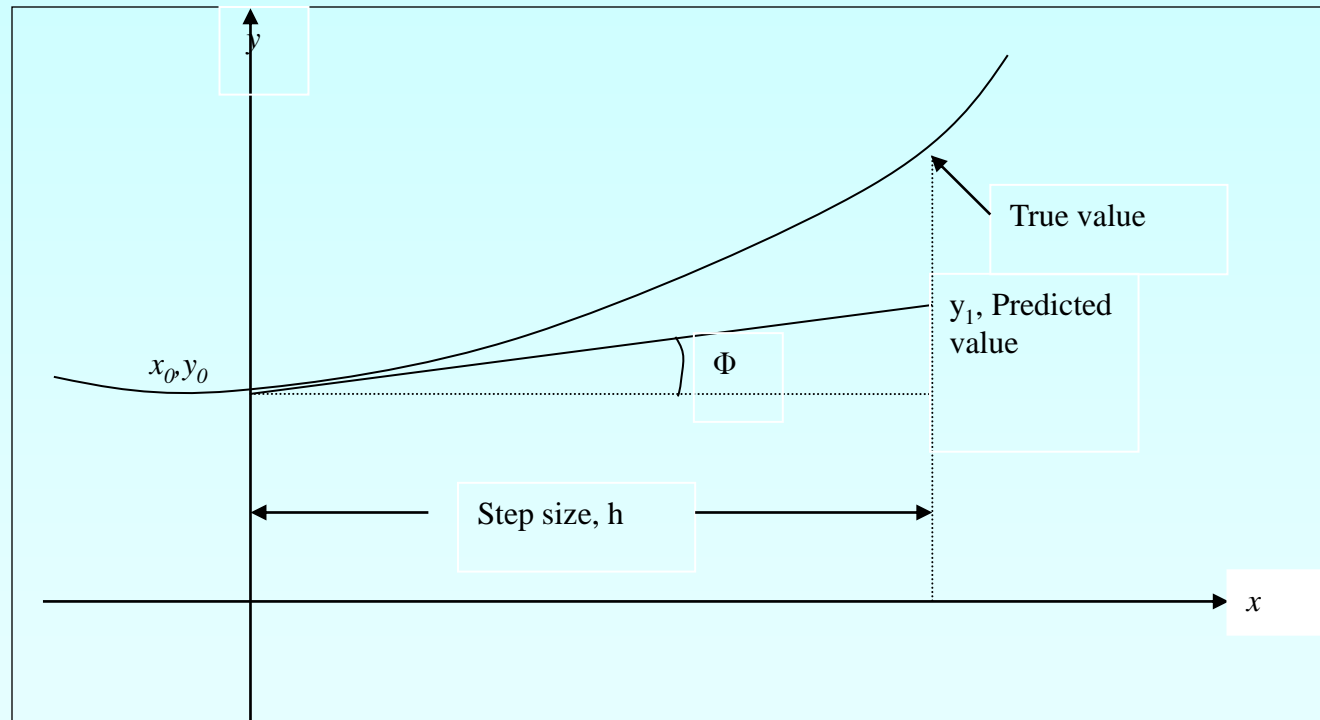
$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

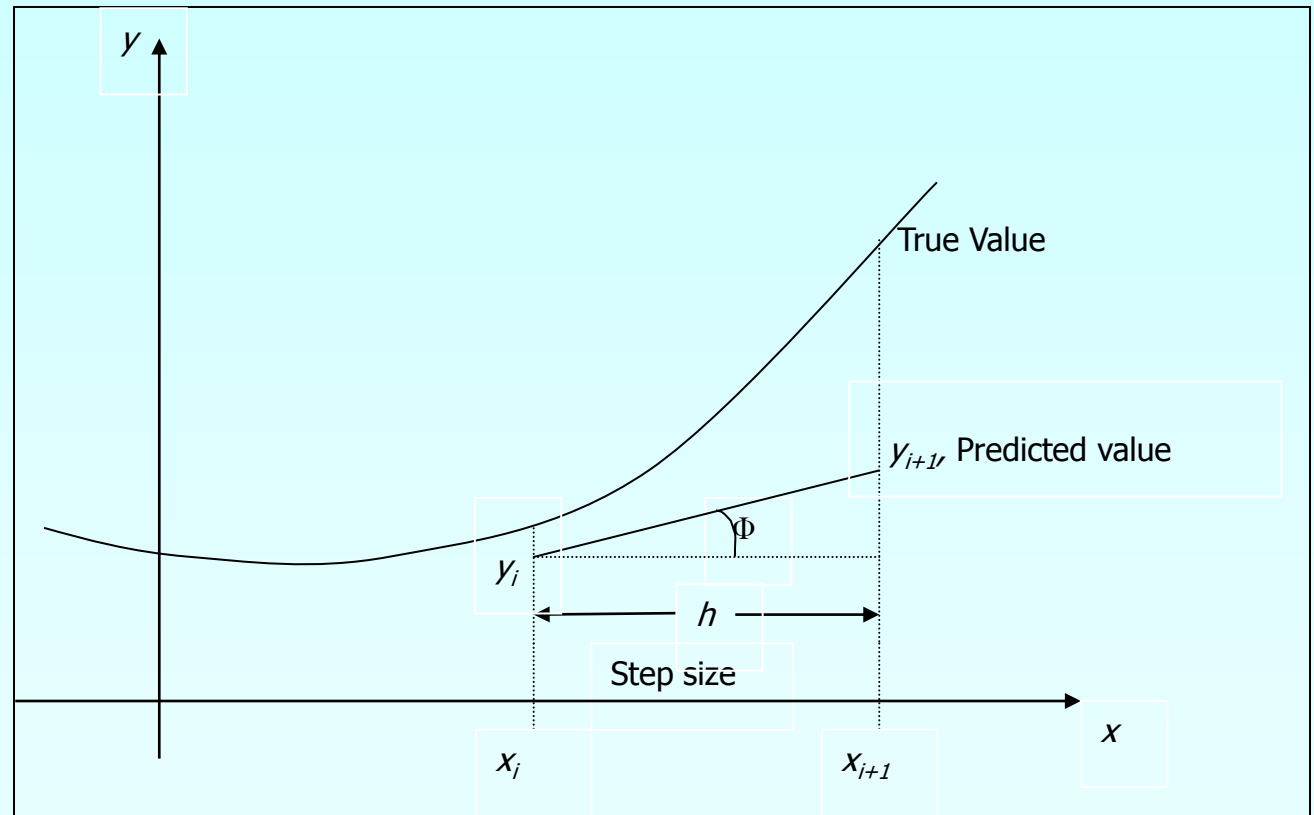


**Figure 1** Graphical interpretation of the first step of Euler's method

# Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$



**Figure 2.** General graphical interpretation of Euler's method

# How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

**Example**

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

# Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at  $t = 480$  seconds using Euler's method. Assume a step size of  $h = 240$  seconds.

# Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

$$f(t, \theta) = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8)$$

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

$$\theta_1 = \theta_0 + f(t_0, \theta_0)h$$

$$= 1200 + f(0, 1200)240$$

$$= 1200 + (-2.2067 \times 10^{-12} (1200^4 - 81 \times 10^8))240$$

$$= 1200 + (-4.5579)240$$

$$= 106.09K$$

$\theta_1$  is the approximate temperature at  $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

# Solution Cont

**Step 2:** For  $i = 1$ ,  $t_1 = 240$ ,  $\theta_1 = 106.09$

$$\begin{aligned}\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\ &= 106.09 + f(240, 106.09)240 \\ &= 106.09 + \left(-2.2067 \times 10^{-12} (106.09^4 - 81 \times 10^8)\right)240 \\ &= 106.09 + (0.017595)240 \\ &= 110.32K\end{aligned}$$

$\theta_2$  is the approximate temperature at  $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

# Solution Cont

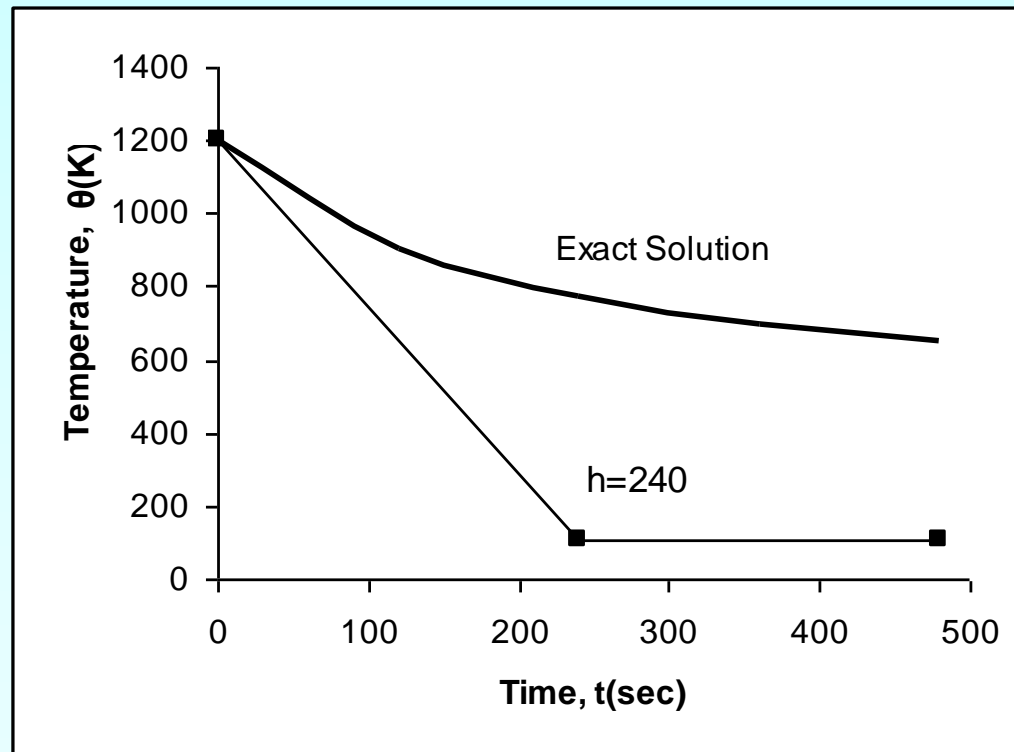
The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at  $t=480$  seconds is

$$\theta(480) = 647.57 K$$

# Comparison of Exact and Numerical Solutions



**Figure 3.** Comparing exact and Euler's method



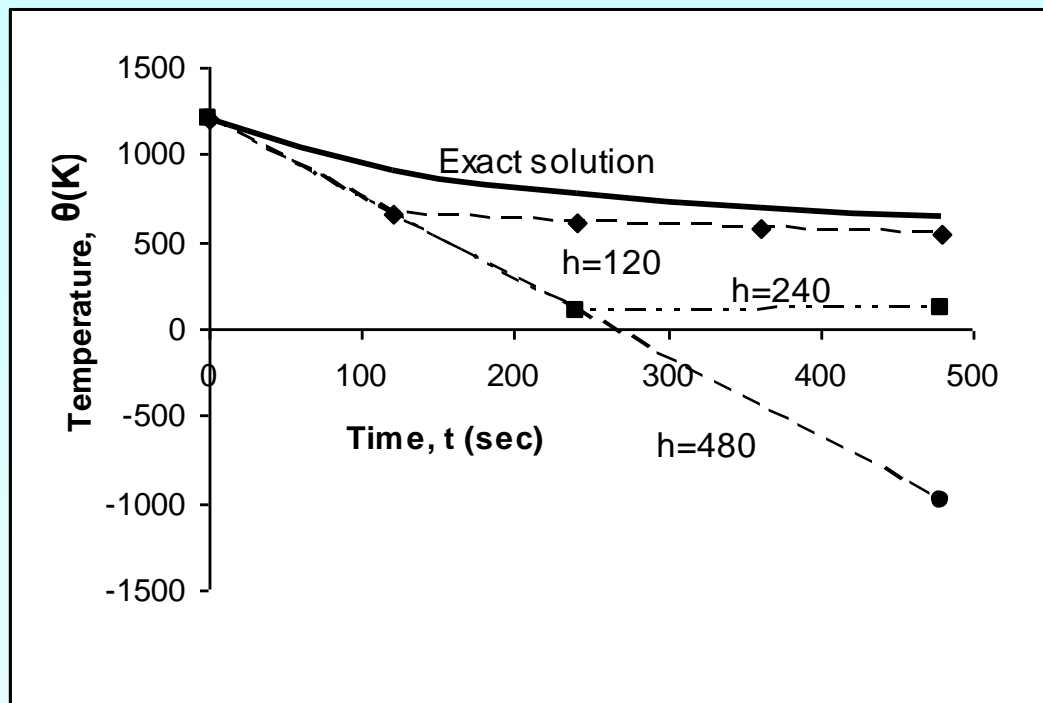
# Effect of step size

**Table 1. Temperature at 480 seconds as a function of step size,  $h$**

Step, $h$	$\theta(480)$	$E_t$	$ \epsilon_t \%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

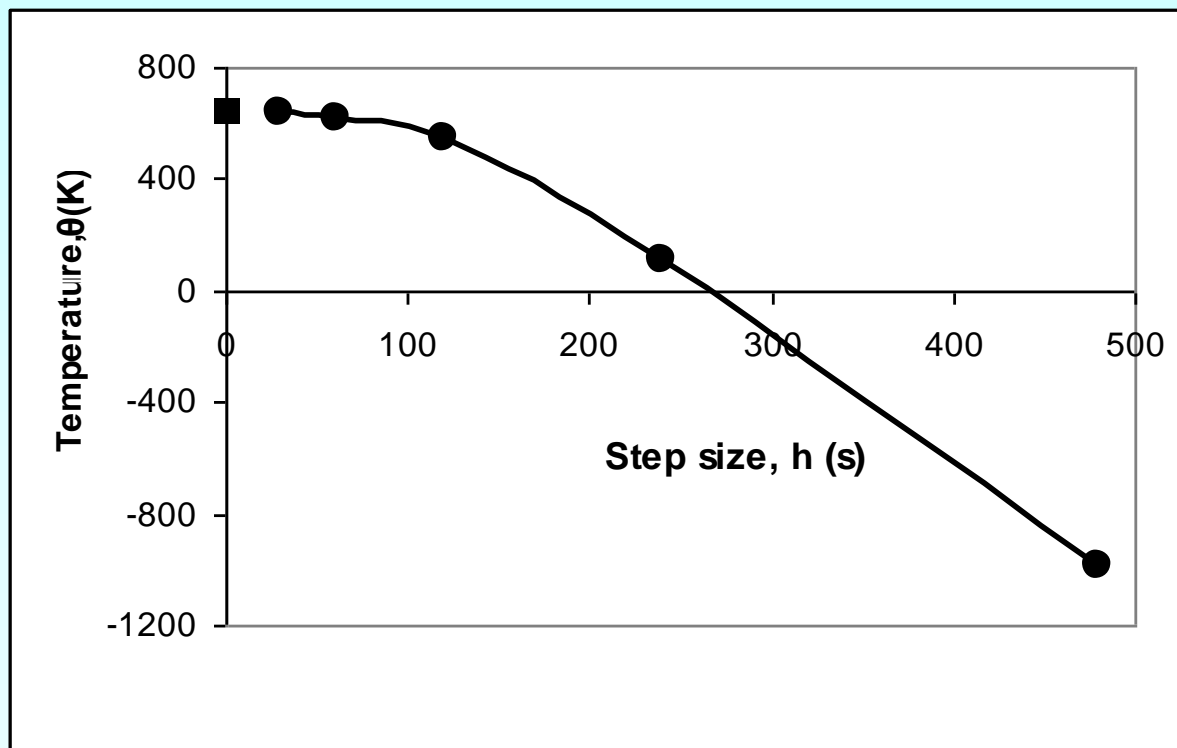
$$\theta(480) = 647.57K \quad (\text{exact})$$

# Comparison with exact results



**Figure 4.** Comparison of Euler's method with exact solution for different step sizes

# Effects of step size on Euler's Method



**Figure 5.** Effect of step size in Euler's method.

# Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left. \frac{dy}{dx} \right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \left. \frac{d^2 y}{dx^2} \right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \left. \frac{d^3 y}{dx^3} \right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \text{are the Euler's method.}$$

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots \quad E_t \propto h^2$$

# Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

[http://numericalmethods.eng.usf.edu/topics/euler\\_method.html](http://numericalmethods.eng.usf.edu/topics/euler_method.html)

**THE END**

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